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Quasilinear approximation for exact coherent states in parallel shear flows

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Abstract

In the quasilinear approximation to the Navier–Stokes equation a minimal set of nonlinearities that is able to maintain turbulent dynamics is kept. For transitional Reynolds numbers, exact coherent structures provide an opportunity for a detailed comparison between full direct numerical solutions of the Navier–Stokes equation with their quasilinear approximation. We show here, for both plane Couette flow and plane Poiseuille flow, that the quasilinear approximation is able to reproduce many properties of exact coherent structures. For higher Reynolds numbers differences in the stability properties and the friction values for the upper branch appear that are connected with a reduction in the number of downstream wavenumbers in the quasilinear approximation. The results show the strengths and limitations of the quasilinear approximation and suggest modelling approaches for turbulent flows.

Keywords: transition, exact coherent structures, quasilinear approximation

1. Introduction

The properties of turbulent flows are determined by the nonlinearities of the Navier–Stokes equations. An analytical treatment of the nonlinearities is usually not possible, and direct numerical simulations that take the full nonlinearities into account are also challenging...
(Cimarelli et al. 2016). For even higher Reynolds numbers approximations such as large eddy simulations or cascade models are needed. More recently, the quasilinear approximation\footnote{Note that in the context of differential equations, partial or ordinary, quasilinear refers to equations where the term with the highest derivative is linear, so that other terms may be nonlinear. By this nomenclature, also the full Navier–Stokes equation is quasilinear since the highest derivatives are contained in the viscous term, which is linear. Here, quasilinear refers to a certain approximation in which parts of the nonlinear interactions are removed, as detailed below.} has attracted attention because of its ability to reproduce certain features in geostrophic flows (Farrell and Ioannou 2007, Tobias and Marston 2013, Constantinou et al. 2016, Marston et al. 2016, Tobias and Marston 2017) as well as in shear flows (Thomas et al. 2014, 2015, Bretheim et al. 2015). Common to all variations of the model is a split of the modes in the Navier–Stokes equation into two groups. The first group contains modes for which all nonlinear couplings are kept. The remaining modes are collected in the second group and for their dynamics, all self-interactions are dropped. Their dynamics then becomes equivalent to a linearization around the flows set up by the first group. To close the loop and introduce a feedback of the second group to the first group, the nonlinearities among the second group that connect to modes in the first group are kept in the dynamics of the first group. This description covers the basic quasilinear and generalised quasilinear approximation, and it shows how to transit from the approximation to the full system by enlarging the set of modes contained in the first group.

Variations of this basic setting introduce stochastic representations of the nonlinear interactions in the second group, in direct statistical simulations (Constantinou et al. 2016, Marston et al. 2016) or stochastic structural stability theory (S3T) (Farrell and Ioannou 2014). Here we will not consider such extensions and work with the most basic model alone.

Previous studies have compared the properties of the full system and the quasilinear approximation for structures (Thomas et al. 2014, 2015, Marston et al. 2016, Nikolaidis et al. 2016, Tobias and Marston 2017) and mean profiles. These studies focus on the statistical properties of the flow and require time-integrations and subsequent averaging to extract mean profiles and stresses. The exact coherent structures that exist in parallel shear flows provide a means to compare the full solution and the approximate solution in a setting where the structures are stationary, either directly as in the case of plane Couette flow, or in a co-moving frame of reference as in plane Poiseuille flow. In particular, this setting allows to explore and compare the bifurcations that lead to the transition to turbulence.

The structure of the paper is as follows. In section 2 we describe the quasilinear approximation for parallel shear flows. In section 3 we study exact coherent structures in plane Couette flow, and in section 4 in plane Poiseuille flow. Concluding remarks are given in section 5.

2. Quasilinear approximation for parallel shear flows

The definition of the quasilinear approximation requires a basis for the velocity fields, such that

\[
\mathbf{u}(\mathbf{x}, t) = \sum_i a_i(t) \mathbf{u}_i(\mathbf{x}),
\]

where we assume that the basis fields \( \mathbf{u}_i(\mathbf{x}) \) are divergence free so that we do not have to worry about the pressure components that assure incompressibility. The set of modes is then split into two groups, one in which all nonlinear interactions are kept, and the other in which the nonlinear interactions within the group are omitted. Introducing projections onto these
(divergence free) subspaces (Frisch 1995), denoted by $\mathcal{P}_1$ and $\mathcal{P}_2$, we can uniquely split any (incompressible) velocity field into two parts if the projections add up to the identity $I$ in the space of incompressible fields,

$$\mathcal{P}_1 + \mathcal{P}_2 = I$$

and do not overlap,

$$\mathcal{P}_1\mathcal{P}_2 = 0.$$  

The quadratic nonlinearity in the Navier–Stokes equation then expands into four terms, each of which can then be projected again onto the two groups, thereby giving rise to eight terms:

$$\begin{align*}
(u \cdot \nabla u) + \nabla p &= \mathcal{P}_1^T(u_1 \cdot \nabla u_1) + \mathcal{P}_1^T(u_2 \cdot \nabla u_2) \\
&+ \mathcal{P}_1^T(u_1 \cdot \nabla u_2) + \mathcal{P}_1^T(u_2 \cdot \nabla u_1) \\
&+ \mathcal{P}_2^T(u_1 \cdot \nabla u_1) + \mathcal{P}_2^T(u_2 \cdot \nabla u_2) \\
&+ \mathcal{P}_2^T(u_1 \cdot \nabla u_2) + \mathcal{P}_2^T(u_2 \cdot \nabla u_1).
\end{align*}$$  

(4)

Here, $\tilde{p}$ denotes the pressure terms that are needed to keep $(u \nabla)u$ incompressible. The Navier–Stokes equation can contain an external pressure term $p$ as a forcing for the flow, as in the case of Poiseuille flows. Using the notation $\mathcal{P}_i = u_i$, the Navier–Stokes equation can be split into two equations, one in each subspace:

$$\begin{align*}
\partial_t u_1 + \mathcal{P}_1^T(u_1 \cdot \nabla u_1) + \mathcal{P}_1^T(u_2 \cdot \nabla u_2) &= - \nabla p + \nu \Delta u_1, \\
\partial_t u_2 + \mathcal{P}_2^T(u_1 \cdot \nabla u_1) + \mathcal{P}_2^T(u_2 \cdot \nabla u_2) &= \nu \Delta u_2.
\end{align*}$$

(5)

This set of equations is still complete and equivalent to the full Navier–Stokes equation.

In the quasilinear approximation, the nonlinear terms

$$\begin{align*}
\mathcal{P}_1^T(u_1 \cdot \nabla u_2), \quad \mathcal{P}_1^T(u_2 \cdot \nabla u_1), \quad \mathcal{P}_2^T(u_1 \cdot \nabla u_3), \quad \mathcal{P}_2^T(u_2 \cdot \nabla u_2)
\end{align*}$$

(7)

are dropped, giving the Navier–Stokes equation in quasilinear approximation

$$\begin{align*}
\partial_t u_1 + \mathcal{P}_1^T(u_1 \cdot \nabla u_1) + \mathcal{P}_1^T(u_2 \cdot \nabla u_2) &= - \nabla p + \nu \Delta u_1, \\
\partial_t u_2 + \mathcal{P}_2^T(u_1 \cdot \nabla u_1) + \mathcal{P}_2^T(u_2 \cdot \nabla u_2) &= \nu \Delta u_2.
\end{align*}$$

(8)

The equation for $u_1$ contains the full nonlinearities and a forcing through the feedback $\mathcal{P}_1^T(u_2 \cdot \nabla u_2)$ from the second group of modes. The equation for $u_2$ is linear in $u_2$ and has the appearance of a Navier–Stokes equation linearised around $u_2$. Moreover, the truncation of the Navier–Stokes equation is such that the nonlinear terms still preserve the energy in both sets. Since an inner product with modes in set $i$ accounts for the projection, products of the type $u_i \cdot \mathcal{P}_i v$ with some arbitrary velocity field $v$ are equal to $u_i \cdot v$. Therefore, the nonlinear terms for the time-dependence of the energy become

$$\begin{align*}
&u_1 \cdot \mathcal{P}_1^T(u_1 \cdot \nabla u_1) + u_2 \cdot \mathcal{P}_2^T(u_1 \cdot \nabla u_2) + u_1 \cdot \mathcal{P}_2(u_2 \cdot \nabla u_2) + u_2 \cdot \mathcal{P}_2(u_2 \cdot \nabla u_1) \\
&= (u_1 \cdot \nabla u_1^2/2 + (u_1 \cdot \nabla u_2^2/2) + (u_2 \cdot \nabla (u_1 \cdot u_2)).
\end{align*}$$

(10)

The last term vanishes since $u_1 \cdot u_2 = 0$, and the first two terms are divergences that do not contribute when integrated over volumes with rigid or periodic boundary conditions. Thus, the nonlinear terms preserve the kinetic energy $(u_1^2 + u_2^2)/2$.

For parallel shear flows, and a splitting into components that are invariant in the flow direction (group 1) and those with a streamwise modulation (group 2), some of the terms
vanish naturally. For instance, interactions within group 1 do not produce a downstream modulation, so that always \( P_2(u_1 \cdot \nabla u_1) = 0 \). Similarly, an interaction between a field with downstream modulation and one without will always keep a downstream modulation, so that the terms \( P_0(u_1 \cdot \nabla u_2) \) and \( P_2(u_1 \cdot \nabla u_2) \) also vanish by construction. Therefore, the quasilinear approximation consists in the elimination of the interaction \( P_2(u_1 \cdot \nabla u_2) \) among the modes in group 2.

The equation for \( u_2 \) is a homogeneous linear equation, parametrically coupled to \( u_1 \). The impact and significance of this parametric feedback is discussed in Farrell and Ioannou (2012). The absence of a forcing by the term \( P_2(u_1 \cdot \nabla u_2) \) in the equation for \( u_2 \) together with the linearity of the equation implies that \( u_2 = 0 \) is a solution. But then there is no forcing on the group 1 modes, and a result by Moffatt (1990) shows that the laminar solution appears as the only translationally invariant solution. Thus, the forcing by the second group of modes is essential.

Nontrivial solutions in the equation for \( u_2 \) require suitable velocity fields \( u_1 \). Specifically, for the case of stationary solutions, the velocity field \( u_1 \) has to be such that the linear operator has a neutral eigenvalue. The corresponding neutral eigenmode then feeds back into the equation for \( u_1 \), and the amplitude of the eigenmode, which is not set by the equation for \( u_2 \), has to be adjusted such that \( u_1 \) gives rise to the neutral eigenmode in the equation for \( u_2 \). For time-dependent flows, this self-consistent feedback is part of the parametric forcing described by Farrell and Ioannou (2012, 2014). Further discussion of this self-consistency requirement will be explored elsewhere, here we simply take the fields and amplitudes as they follow from the numerical solutions, typically by continuing the fully nonlinear solution to the quasilinear approximation. To this end, we introduce the interactions \( P_2(u_1 \cdot \nabla u_2) \) among the modes in group 2 (recall that the other terms disappear naturally) with a parameter \( q \) which we can tune from \( q = 1 \) corresponding to the full Navier–Stokes equation to \( q = 0 \) for the quasilinear approximation.

For the comparison between exact coherent states in the full system and in the quasilinear approximation, we turn to plane Couette flow and plane Poiseuille flow in domains that are periodic in the downstream (\( x \)-) and spanwise (\( z \)-) direction. Accordingly, the velocity fields can be expanded in Fourier modes in these two directions. Group 1 contains all modes that do not have a dependence on the downstream direction, i.e. \( k_z = 0 \). Likewise, group 2 contains all modes with \( k_z \neq 0 \).

The decomposition in Fourier modes reveals a degeneracy that is intrinsic to the quasilinear approximation. Since Fourier modes are orthogonal, each Fourier mode gives an isolated contribution to the modes in the first set. Similarly, the equation in the second group separates into individual equations for each Fourier component. Moreover, the equations are linear, so that not only the amplitude but also the phases of the Fourier modes in group 2 are free parameters. In the contribution to the equations for the translationally invariant modes in group 1, only the absolute value squared enters, so that the phases vanish. Accordingly, in the quasilinear approximation the phases between the modes in group 2 are not fixed. All modes in group 2 can be shifted arbitrarily and independent of each other in the downstream direction, without affecting \( u_1 \). As a consequence, if simulations are started in the quasilinear approximation, the phases are arbitrary, which can give rise to velocity fields that look unusual because they are not constrained by the nonlinear interactions. However, when the quasilinear approximation is obtained by continuation from the full equations suitable phases are inherited and the velocity fields become reasonable. In the calculations that are presented below, this problem does not occur and the degeneracy is not visible because only modes with \( k_z = \pm 1 \) appear. The degeneracy could only be observed if a second set of modes with \( k_z = \pm 2 \) or similar was excited. Then meaningful velocity fields can only be obtained by
continuation from the fully nonlinear setting in which the relative phases between the different modes are fixed.

3. Quasilinear exact coherent states for plane Couette flow

Exact coherent states and their quasilinear approximation are computed within the channel-flow programme suite (Gibson 2010) with a Fourier–Chebychev–Fourier spectral method. With $x$, $y$, and $z$ the downstream, normal and spanwise direction, the domain is $L_x = 2\pi/\alpha$ and $L_z = 2\pi/\gamma$ wide, with $\alpha = 0.577$ and $\gamma = 1.150$, and $L_y$ high. We used a numerical resolution of $48 \times 33 \times 48$ modes, for which convergence was verified for Reynolds numbers up to $Re = 1000$. The fully nonlinear state at Reynolds number 400 coincides with the lower and upper branch states EQ1 and EQ2, respectively, described by Nagata (1990), Clever and Busse (1997), Waleffe (2003), Gibson et al (2009) and the results in this section have been obtained from computations within the symmetry subspace of these states.

On the left-hand side of figure 1 the bifurcation diagram in the optimal domain for EQ1, as determined in (Waleffe 2003), is shown. The critical values for the full nonlinear state are $Re_c = 127.7$ and $D_c = 1.8$, whereas the values for the quasilinear approximation are $Re_c = 135.5$ and $D_c = 1.75$, in very good agreement. The lower branches stay close together at least up to Reynolds numbers $Re = 750$, whereas the upper branches move apart for $Re > 300$. Similar behaviour is found for the states EQ7 and EQ8. Also for these states the bifurcation branches stay close together and have critical points at $Re_c = 234.1$ and $D_c = 2.27$ in the full system and $Re_c = 227.3$ and $D_c = 2.07$ in the quasilinear approximation. The upper branches share the initial rapid increase in friction, a separation of the branches cannot be examined as EQ8 does not exist for higher $Re$.

The flow fields for EQ1 and EQ2 at $Re = 400$ are shown in figure 2. The lower branch states in the full and quasilinear spaces are essentially indistinguishable, and the upper branch states differ in particular in the sideways modulation, where the state from the full simulations has more structure and a more pronounced amplitude.
The differences between the states also show up in their modal content, see figure 3. The lower branch state in the full simulation has contributions from a wider range of wavenumbers in the spanwise direction $k_z$, but only over a very narrow range in the downstream direction $k_x$. Over a set of wavenumbers from $k_x = 1$ to $k_x = 4$, the amplitude of the modes drops by four decades. The quasilinear state shows about the same range of modes and amplitudes in the spanwise direction, but only has modes $k_x = 0$ and $\pm 1$ in the downstream direction. Nevertheless, the differences between a rapid drop in $k_x$ or a severe restriction in $k_x$ modify the flow field and the bifurcation diagram only very little.

For the upper branch state, the differences are more pronounced: the full state has a significant contribution from downstream wavenumbers and the amplitude drops more slowly with $k_x$, covering a range of $k_x = 1$ to $k_x = 13$ before the amplitude is down by four decades. In contrast, the quasilinear state has only 3 downstream wavenumbers (just as for the lower branch state) which reflects the absence of finer modulations in the upper branch. It is remarkable, however, that the differences in dissipation are not as large, see the bifurcation diagram, figure 1.

The differences in the Fourier modes also affect the stability properties of the states, see figure 4. For the lower branch at $Re = 400$, the eigenvalue spectra of the full and quasilinear state are very similar. They both have one positive eigenvalue $0.0315$ and $0.0353$, respectively, representing the one unstable direction of the lower branch state. The next eigenvalues are $-0.0016$ for the full state and $-0.0012$ for the quasilinear state. Also the overall distribution of the eigenvalues as well as the range in imaginary parts are similar.

For the upper branch at $Re = 400$, the situation is very different: the full state has 35 positive eigenvalues, 1 real eigenvalue and 17 pairs of complex eigenvalues. In contrast, the quasilinear state has only 10 positive eigenvalues which come in pairs of complex conjugate eigenvalues. The maximal eigenvalues in the quasilinear approximation are also much smaller than for the fully nonlinear state. Apparently the additional modes in $k_x$ drive further and stronger instabilities. This difference between the quasilinear and full state becomes more
pronounced with increasing Reynolds number, as shown in figure 5. For the full system, the number of unstable eigendirections increases monotonically with $Re$, whereas for the quasilinear system the variation is non-monotonic, and the rate of increase for higher $Re$ is smaller, most likely because of the absence of higher $k_x$ modes.

In the full nonlinear system, the upper branch state undergoes a sequence of bifurcations that give rise to complex temporal dynamics (Mellibovsky and Eckhardt 2011, 2012, Kreilos and Eckhardt 2012, Avila et al 2013, Zammert and Eckhardt 2015). In figure 6 we compare the bifurcation diagrams obtained in the full space and in the quasilinear approximation for the case studied in Kreilos and Eckhardt (2012). The bifurcation diagram is obtained from the maxima in the time series of the energy in the spanwise and normal velocity components: if the state is a fixed point or a travelling wave, this gives only a single value. Two values indicate a period-doubled state, and a band of values indicates chaotic dynamics. The diagram shows that the dynamics in the quasilinear approximation follows that in the full nonlinear system by passing through a Hopf-bifurcation and a period doubling sequence to chaotic dynamics. The main difference is that the Reynolds numbers

![Figure 3. Modal content of the coherent states at $Re = 400$. Shown are the upper branch (top row) and lower branch (lower row), obtained in the full Navier–Stokes equation (left column) and in the quasilinear approximation (right column). One notes that in the full space, the upper branch is supported by many more modes than the lower branch. In the quasilinear approximation both states contain only $k_x = 0$ and $\pm 1$ modes whose amplitudes are similar to the ones from the full equation.](image-url)
at which the bifurcations take place are higher and that the variations in amplitude are smaller.

Finally, we address the question of the nonlinear recovery by tuning the parameter $q$. For $q$ slightly above 0, one can imagine some kind of perturbation theory that will allow the state to be recovered. With increasing parameter $q$ more and more modes become active and there is a smooth transition from the quasilinear to the full nonlinear state as can be seen in figure 7. The left and right plots show the logarithm of the Fourier amplitudes.

**Figure 4.** Eigenvalues of the exact coherent states at $Re = 400$. The upper branch and lower branch solutions are shown in the top and bottom rows, respectively, and the columns distinguish the eigenvalues in the full space (left column, blue symbols) and in the quasilinear system (right column, red symbols).
independence on $k_x$ and $k_z$ and the sum of the absolute values of the Chebyshev coefficients for a few of the lowest order combinations of $(k_x, k_z)$ on the diagonal, i.e. $(0, 0), (1, 1), \ldots$, respectively. As the parameter $q$ becomes smaller, the modal content is reduced until most of
the modes eventually vanish for \( q = 0 \). For the quasilinear approximation only a few modes contribute to the flow field.

The transition between the quasilinear and the fully nonlinear state can also be seen in the energy content in all modes with a downstream modulation (cross flow energy). These modes are essential for maintaining the turbulent dynamics. As figure 8 shows, the differences remain smaller than 10\% in energy up to \( q \) about 0.5, and are within 10\% of the full value only when \( q \) is above 0.8. The contributions from the different modes is non-uniform in \( q \), as shown in figure 7. Nevertheless, it seems that quantitative agreement requires substantial contributions from the nonlinear interactions.
In the case of plane Poiseuille flow, we compare the representation of two travelling waves, studied by Waleffe (2001) and Hwang et al (2016). The computations are done using the diablo code, in which the streamwise and spanwise directions are discretized using Fourier series, while the wall-normal direction is discretized using the second-order central difference. The code also implements the Newton–Krylov–Hookstep (Viswanath 2007, 2009) as used in (Willis et al 2013). We note that both of the travelling waves here satisfy the mirror symmetry about the mid-plane of the channel and the shift-reflect symmetry. Therefore, all the calculations in this section have been done in the subspace of these symmetries in a computational domain of \((L_x, L_y, L_z) = (3.0, 2.0, 1.5)\). The resolution for Waleffe’s solution is \((N_x, N_y, N_z) = (48, 81, 48)\), and that for Hwang’s solution is \((N_x, N_y, N_z) = (72, 81, 72)\).

The bifurcation diagrams for the two exact coherent structures in the full system and in the quasilinear approximation are shown in figure 9. Here again, the states in the quasilinear approximation satisfy the mirror and shift-reflect symmetries. The bifurcation diagrams show the energy contained in all modes with downstream modulation (cross flow energy) during the transition from the quasilinear solution at \(q = 0\) to the fully nonlinear one at \(q = 1\) for the case shown in figure 7.
approximation are obtained by continuation from the exact solutions. As in the case of plane Couette flow, the bifurcation curves are reasonably well captured by the quasilinear approximation. In particular, the lower-branch solution in Hwang et al. (2016) shows fairly good agreement compared to the other solutions.

Images of the solutions are shown in figure 10. The quasilinear approximation reproduces many of the prominent features of the full solutions quite well, even though it again has only a single nonzero downstream wavenumber. Here, it is also interesting to note that the quasilinear approximation of the upper-branch solutions from Waleffe (2001) (top image in the right column) and in Hwang et al. (2016) (third image in the right column) are fairly similar to each other, despite the fact that the approximation was made from two different solutions.

Finally, the turbulent statistics between the full and quasilinear solutions are compared, as shown in figure 11. Here, the superscript + denotes scaling of the variables using the friction velocity $u_\tau$ and the viscous inner length scale $\delta_v = \nu/u_\tau$ ($\nu$ is the kinematic viscosity). As expected, the good agreement between the full and quasilinear solutions also shows up in the turbulent statistics, especially for the lower-branch case. The contribution from additional downstream wavenumbers increases the spanwise fluctuations, giving rise to the larger deviations in $w_{rms}$. However, the effect on the Reynolds stress $\langle uv \rangle$ is smaller, consistent with the small differences in the mean profiles.

5. Conclusions

The comparison between the quasilinear approximation and the full nonlinear solutions for exact coherent states in plane Couette and plane Poiseuille flow shows that the approximation does remarkably well in reproducing features of the exact coherent states: the bifurcation points are very close together, the subsequent separation between the upper and lower branch states is similar, though there are differences in the upper branch for higher Reynolds numbers.

As noted in previous studies, the quasilinear approximation has a tendency to reduce the number of modes with downstream modulation. In the case of plane Couette flow, the quasilinear approximation contains only a single downstream wavenumber $k_x = \pm 1$ in addition to $k_x = 0$ for Reynolds numbers up to at least about 1000. Nevertheless, even though the full solution covers a wider range in wave numbers, this has little effect on the drag and on the turbulent statistics. This suggests that only a few downstream modes go a long way in maintaining and explaining turbulence in these flows. Eventually, for even higher Reynolds numbers, we do expect some of the higher $k_x$ modes to reappear, as they are present in the simulations of Farrell et al. (2017).

The observation that only a few $k_x$ modes are present in the quasilinear approximation establishes a connection to the low-dimensional models of Waleffe (1997) and Moehlis et al. (2004), which also have only a single downstream wave number. Moreover, the models can also be split into two groups, with a first group of nonlinearly coupled modes (specifically the downstream vortices and streaks, as well as the mean flow and its modulations) and a second set of modes that are parametrically coupled to the nonlinear modes. An analysis of the parametric instabilities in the low-dimensional models could therefore also provide insight into the stability properties of the quasilinear approximation in general. An example of this is provided by the bifurcation diagram shown in figure 6: in both the full and the quasilinear system the exact coherent states that form in saddle-node bifurcations become unstable and
Figure 10. Full (left column) and quasilinear solutions (right column) for the upper and lower branch of the solutions in Waleffe (2001) (top two rows) and in Hwang et al (2016) (lower two rows). Here, the blue iso-surfaces indicate $u^+ = -2$ where $u$ indicates the streamwise velocity fluctuation, and the red iso-surfaces indicate $\lambda_2 = -0.002$ (Jeong and Hussain 1995).
develop into temporally complex chaotic solutions through a sequence of secondary bifurcations.

For the transitional Reynolds numbers studied here the restriction in downstream wavenumbers to $k_x = 0$ and ±1 that comes out naturally within the quasilinear approximation does not result in very big differences between the quasilinear and the full dynamics. The situation may be different for higher Reynolds numbers, and it will be interesting to see whether additional wave numbers are only excited when downstream couplings are introduced (as in the generalised quasilinear approximation) or whether they can also be excited within the quasilinear approximation.

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